Path integral formulation of QFT (II)

- Sredniki : 6, 7, 8, 9
- Schwartz : 14
- P&S : 9

We will discuss now perturbative results using the path integral approach. We split the dynamics in two parts: a part that we can solve and another that we treat perturbatively.
Here, solving means that we know how to compute the path integral. If the action is quadratic, we know the solution, for instance.

The Euclidean action is

$$S = S_0 + S_{int}$$

with S_0 being quadratic,

$$S_0 = \int d^4x\, d^4y\ \frac{1}{2}\, \varphi(x)\, K_{xy}\, \varphi(y)$$

with K_{xy} being a functional acting on the fields, so may contain derivatives.

While all terms of S_0 are quadratic, not all quadratic terms of S are necessarily on S_0.

The theories considered will be local, namely

$$K_{xy} = \delta^4(x-y) \cdot K_y$$

For instance, for a massive scalar field,

$$S_0 = \frac{1}{2} \int d^4x \left(\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2 \right) \quad \text{(Eucl.)}$$

$$= \int d^4x \; \varphi(x) \cdot \frac{1}{2} (m^2 - \Box) \varphi(x)$$

$$\rightarrow \quad K_y = m^2 - \Box_y$$

■ GENERATING FUNCTIONALS & FREE THEORY

We can compute the path integral for S_0. In fact we can compute any correlator by considering a modified path integral where the field $\phi(x)$ is sourced by $J(x)$. This defines the generating functional

$$Z_0[J(\cdot)] \equiv \int D\varphi \; e^{-S_0 + \int d^4x \, J(x) \phi(x)}$$

We can see that this generates all correlators of our theory,

$$\frac{1}{Z_0[J=0]}\left(\frac{\partial}{\partial J(x_1)}\cdots\frac{\partial}{\partial J(x_n)} Z_0[J]\right)\Bigg|_{J=0} = \frac{\int D\varphi\, \varphi_1\cdots\varphi_n\, e^{-S_0}}{\int D\varphi\, e^{-S_0}}$$

We can compute the generating functional $Z_0[J]$ by recalling the gaussian integral

$$\int d\vec{z}\; e^{-\frac{1}{2}\vec{z}^T A\vec{z} + \vec{b}^T\cdot\vec{z}} =$$

$$= \int d\vec{z}\; e^{-\frac{1}{2}(\vec{z}^T - \vec{b}^T A^{-1}).A.(\vec{z}-A^{-1}\vec{b}) + \frac{1}{2}\vec{b}^T.A^{-1}.\vec{b}}$$

$$= e^{\frac{1}{2}\vec{b}^T.A^{-1}.\vec{b}} \int d\vec{z}'\, e^{-\frac{1}{2}\vec{z}'^T.A.\vec{z}'}$$

$$= e^{\frac{1}{2}\vec{b}^T.A^{-1}.\vec{b}} \cdot \sqrt{\frac{\pi}{\det A}}$$

This requires A to be a positive definite matrix. In our case, we have that A is the functional of the quadratic term and $\vec{b}$ is the source $J(x)$:

$$Z_0[J] = e^{\frac{1}{2}\int dx\, dy\; J(x) K^{-1}_{xy} J(y)} \sqrt{\frac{\pi}{\det K}}$$

$$= \underbrace{Z_0[J=0]}_{\text{irrelevant constant}} \times e^{\frac{1}{2}\int dx\, dy\; J(x) D_{xy} J(y)}$$

Here D_{xy} is the inverse of the Kinetic operator,

$$\int d^4z \, K_{xz} D_{zy} = \delta^4(x-y)$$

Note that K must be a positive functional, so all eigenfunctions have positive eigenvalues. If this is not the case, directions in field space will be unbounded and give divergent contributions to the path integral. This would mean that S_0 does not correctly capture the dynamics and other terms in S are needed to stabilize the path integral.

- The D_{xy} functional is the Euclidean Feynman propagator, since it is the two point function:

$$\frac{\int D\varphi \, \varphi(x_1)\varphi(x_2) \, e^{-S_0}}{\int D\varphi \, e^{-S_0}} = \frac{1}{Z_0[J=0]} \left. \frac{\partial^2 Z_0[J]}{\partial J(x_1) \partial J(x_2)} \right|_{J=0}$$

$$= D_{x_1 x_2}$$

For a local K,

$$K_{xy} = \delta^4(x-y)\, K_y$$

the equation for D is

$$\int dz\, K_{xz} D_{zy} = K_x D_{xy} = \delta^4(x-y)$$

We can express D in Fourier space, with the convention

$$f(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \tilde{f}(p)$$

$$\tilde{f}(p) = \int d^4x\, e^{ip\cdot x} f(x)$$

(The sign in $e^{-ip\cdot x}$ is the standard one even if we are in Euclidean. We will continue p_μ with lower indices, so $p_0 \to -ip_0$, $p_i \to p_i$; so since $x^0 \to ix^0$, $x^i \to x^i$ we have $p\cdot x \to p\cdot x$)

D_{xy} will only depend on $x-y$, so we write

$$D_{xy} = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \tilde{D}(p).$$

By direct substitution,

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, \tilde{D}(p)\, K_x\left(e^{-ip\cdot(x-y)}\right)$$

Since K_x is a local differential operator, its action on plane waves is diagonal,

$$K_x\left(e^{-ip(x-y)}\right) = e^{-ip(x-y)}\, \tilde{K}(p)$$

with $\tilde{K}(p)$ being some polynomial in p. We get

$$\tilde{D}(p) = \frac{1}{\tilde{K}(p)}$$

and

$$D_{xy} = \underbrace{\Delta_F^{(E)}(x-y)}_{\text{Euc. Feyn. prop.}} = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{1}{\tilde{K}(p)}$$

So in Fourier space the propagator is the inverse of the kinetic term.

In the case of the scalar field,

$$K_x = m^2 - \Box_x \longrightarrow \tilde{K}(p) = p^2 + m^2$$

So the propagator

$$\Delta_F^{(E)} = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{1}{p^2+m^2}$$

• At this point, the free theory can be considered as solved. We can compute any correlator, for instance:

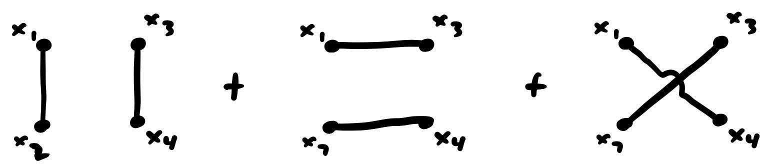
$$\frac{1}{Z}\frac{\partial^4 Z}{\partial J(x_1)\partial J(x_2)\partial J(x_3)\partial J(x_4)}\bigg|_{J=0} = \frac{1}{Z}\frac{\partial^3 Z}{\partial J_1 \partial J_2 \partial J_3} \times$$

$$\times \left(\int d^4z\, D_{x_4 z} J(z)\right) e^{\frac{1}{2}\int dx dy J_x D_{xy} J_y}$$

$$= \frac{\partial^3 Z}{\partial J_1 \partial J_2 \partial J_3} (J_z D_{z4})\, e^{\frac{1}{2} J_x D_{xy} J_y}$$

$$= \frac{\partial^2 Z}{\partial J_1 \partial J_2} (D_{34} + J_z D_{z3} + J_w D_{w4})\, e^{\frac{1}{2} J_x D_{xy} J_y}$$

$$= \ldots = D_{34} D_{12} + D_{23} D_{14} + D_{13} D_{24}$$

$=$ 

These are the same cotractions one finds in the canonical approach.

• What we did generalizes to theories with any (bosonic) field content. Be $\Phi_M(x)$ a vector whose components are the fields in the theory. The general action is

$$S_0 = \int d^4x\, d^4y\, \frac{1}{2}\, \phi_M(x)\, K_{\{M,x\}\{N,y\}}\, \phi_N(y)$$

We find exactly as before,

$$Z_0[J] = Z_0[0]\, e^{\frac{1}{2}\int dx\, dy\, J_M(x)\, D_{\{M,x\}\{N,y\}}\, J_N(y)}$$

where we have now external sources J_M, one for each field. D is defined by

$$\int d^4z\, K_{\{M,x\}\{R,z\}}\, D_{\{R,z\}\{N,y\}} = \delta_{MN}\, \delta^4(x-y)$$

When K is local,

$$K_{\{M,x\}\{N,y\}} = \delta^4(x-y)\, (K_y)_{MN}$$

and in Fourier space

$$(K_x)_{MN}\left[e^{-ip\cdot(x-y)}\right] = (\tilde{K}(p))_{MN}\, e^{-ip\cdot(x-y)}$$

We have that

$$(K_x)_{MR}\, D_{\{R,x\}\{N,y\}} = \delta_{MN}\, \delta^4(x-y)$$

and

$$D_{\{M,x\}\{N,y\}} = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, (\tilde{D}(p))_{MN}$$

and therefore

$$(\tilde{K}(p))_{MR}\, (\tilde{D}(p))_{RN} = \delta_{MN} \quad \Rightarrow \quad \tilde{D} = \tilde{K}^{-1}$$

Again, the propagator is the inverse of the kinetic matrix in Fourier space.
As before, D is the two-point function.

- We solved the quadratic theory by computing $Z_0[J]$ in full generality.
  We can already appreciate the advantage with respect the canonical formalism, since we can solve any quadratic lagrangian.

■ TO MINKOWSKI

Before considering the interacting theory, we should remember that we solved the theory in Euclidean. We should continue our results to Minkowski space.

Consider the single field case, but with $\tilde{K}(p)$ generic, so $\tilde{K}(p)$ is a polynomial in p.

The Euclidean propagator is

$$\Delta_F^{(E)}(x) = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{x}} \int \frac{dp_0}{2\pi}\, e^{-ip_0 x^0} \frac{1}{\tilde{K}(p_0;\vec{p})}$$

The p_0 integral is the interesting part.
We write it as an integral of a complex variable z along the real line,

$$I_E(x^0) = \frac{1}{2\pi}\int_{z\in\mathbb{R}} dz\, \frac{e^{-ix^0 z}}{\tilde{K}(z)}$$

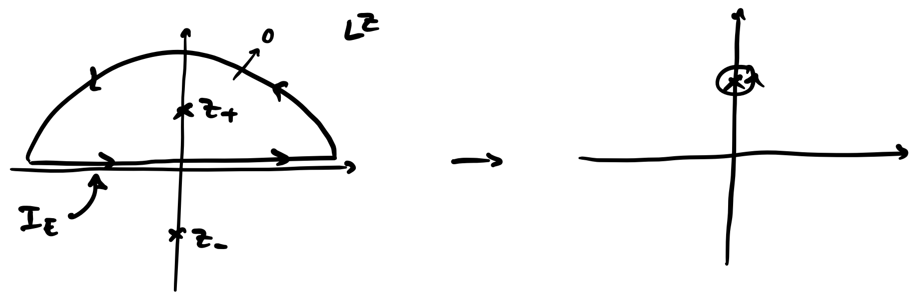
The only poles of the integrand are those of $1/\tilde{K}$. For the case of a scalar field,

$$\frac{1}{\tilde{K}} = \frac{1}{z^2+E^2} = \frac{1}{(z-iE)(z+iE)}$$

with $E = \sqrt{|\vec{p}|^2 + m^2}$. The poles are at $z_\pm = \pm i E$, so in the imaginary axis.

Lorentz symmetry requires p^2 terms, so even in the generic case the poles are paired. In no case we will have poles on the real Euclidean line.

Take the case of two poles, and $x^0 < 0$. If $x^0 > 0$ the argument is similar. Closing the contour in the upper half plane,



$$I_E = i \operatorname*{Res}_{z_+} \left(\frac{e^{-ix^0 z}}{\tilde{K}(z)} \right) = i e^{-ix^0 z_+} \operatorname*{Res}_{z_+} \left(\frac{1}{\tilde{K}(z)} \right)$$

$$= \frac{1}{2E} e^{E x^0} = \frac{1}{2E} e^{-E|x^0|}$$

The propagator is exponentially suppressed at large distance in the Euclidean.

• In order to obtain the Minkowski propagator, we need to replace $x^0 \to i x^0$,

$$I_M(x^0) = I_E(ix^0) = i\, e^{+x^0 z_+} \, \mathrm{Res}\left(\frac{1}{\tilde{K}}\right)$$

$$= \frac{1}{2E} e^{iEx^0}$$

The real-time propagator, for $x^0 < 0$, is

$$\Delta_F(x) = \Delta_F^{(E)}(ix^0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} I_M(x^0)$$

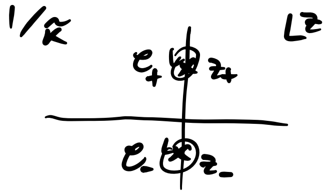
$$= \int \frac{d^3p}{(2\pi)^3} e^{iEx^0} e^{-i\vec{p}\cdot\vec{x}} \frac{1}{2E_{\vec{p}}}$$

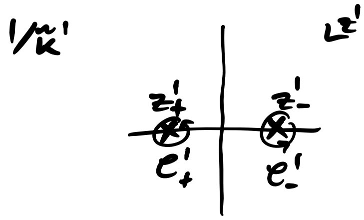
So we recovered the standard result of the scalar case. The propagator can be written as the d^4p integral on the "Feynman integration contour". This is possible in general. Define a function $\tilde{K}'(z')$,

$$\tilde{K}'(z') \equiv \tilde{K}(-iz')$$

so obtained by substituting $z \to -iz$ in the original. This amounts to $p_0 \to -ip_0$ to "continue"

Euclidean momentum to Minkowski. The zeroes of $\tilde{K}'$ are at $z'_\pm = i z_\pm$, so the residues of $1/\tilde{K}$ are on the real axis.





The residues are related by

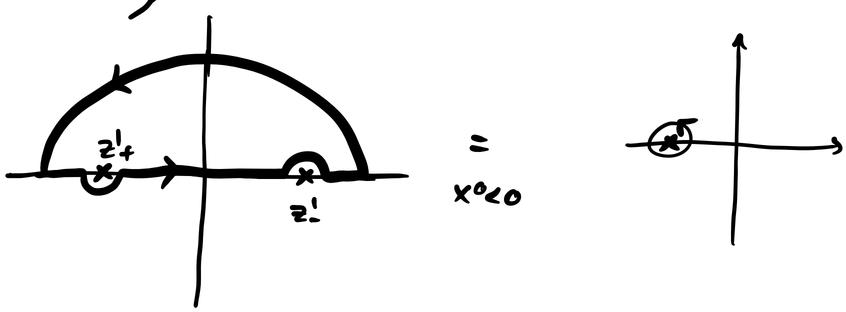
$$2\pi i \operatorname*{Res}_{z'_+}\left(\frac{1}{\tilde{K}'}\right) = \int_{\mathcal{C}'_+} dz' \frac{1}{\tilde{K}'(z')} = \int_{\mathcal{C}'_+} dz' \frac{1}{\tilde{K}(-iz')} = i\int_{\mathcal{C}_+} dz \frac{1}{\tilde{K}(z)}$$

$$\to \operatorname*{Res}_{z'_\pm}\left(\frac{1}{\tilde{K}'}\right) = i \operatorname*{Res}_{z_\pm}\left(\frac{1}{\tilde{K}}\right)$$

- Consider then the integral

$$-i\int_{F.C.} \frac{dz'}{2\pi}\, e^{-ix^0 z'} \frac{1}{\tilde{K}'(z')}$$

on the "Feynman contour"



so that

$$\int \frac{dz'}{2\pi} \cdots = i\, e^{-ix^0 z'} \operatorname*{Res}_{z'_+}\left(\frac{1}{\tilde{K}}\right)$$

$$= i\, e^{+x^0 z_+} (+i) \operatorname*{Res}_{z_+}\left(\frac{1}{\tilde{K}}\right)$$

So $I_M(x^0)$, for $x^0 < 0$, is equal to

$$I_M(x^0) = i\, e^{x^0 z_+} \operatorname{Res}\left(\frac{1}{\tilde{K}}\right) = -i \int_{F.C.} \frac{dz'}{2\pi}\, e^{-ix^0 z'} \frac{1}{\tilde{K}(z')}$$

This leads to the general rule to go from Euclidean quantities expressed as momentum integrals to Minkowski quantities expressed as integrals over the Feynman contour.

$$\begin{cases} p_0^{(E)} = -i\, p_0 \\ p_i^{(E)} = p_i \\ p^{i(E)} = -p^i \end{cases}, \quad \begin{cases} x_E^0 = i x^0 \\ x_E^i = x^i \\ x_{E,i} = -x_i \end{cases} \Rightarrow \quad \begin{matrix} p_E^2 \to -p^2 \\ p \cdot x \to +p \cdot x \end{matrix}$$

In the case of the scalar,

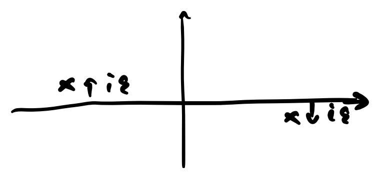
$$\Delta_F^{(E)}(x_E) = \int \frac{d^4 p_E}{(2\pi)^4}\, e^{-ip_E \cdot x_E} \frac{1}{p_E^2 + m^2}$$

$$\rightarrow \quad \Delta_F(x) = (-i)\int_{F.C.} \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2+m^2} e^{-ip\cdot x}$$

$$= \int_{F.C.} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{p^2-m^2}$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{p^2-m^2+i\varepsilon}$$

The $+i\varepsilon$ is equivalent to the F.C. integration:



INTERACTION VERTICES

Suppose you want to get correlators in the theory

$$S = S_0 + S_{int}$$

so we must perform integrals like

$$\int D\varphi \; \varphi(x_1)\cdots\varphi(x_n)\; e^{-S_0 - S_{int}}$$

where the interaction part is of the type

$$S_{int} = \int d^4x\, \lambda\phi^4 \;, \text{ or } \int d^4x\, \phi^3\Box\phi \;, \text{ etc...}$$

Perturbation theory consists of expanding the interacting part as

$$\int D\varphi \, \varphi \cdots \varphi \, e^{-S_0} \sum_k \frac{1}{k!} (-S_{int})^k$$

However, since S_{int} is a functional of fields and derivatives, we know how to evaluate these type of integrals.

· For instance,

$$\int D\varphi \, \varphi \cdots \varphi \, (-S_{int})^k e^{-S_0} = \langle 0 | T\{ \varphi \cdots \varphi \, (-S_{int})^k \} | 0 \rangle \times \underbrace{\int D\varphi \, e^{-S_0}}_{\text{norm.}}$$

Therefore the Euclidean correlators in the interacting theory are

$$\langle 0 | T\{ \varphi(x_1) \cdots \varphi(x_n) \} | 0 \rangle_E = \frac{\int D\varphi \, e^{-S_0 - S_{int}} \varphi \cdots \varphi}{\int D\varphi \, e^{-S_0 - S_{int}}}$$

$$= \frac{\langle 0 | T\{ \varphi(x_1) \cdots \varphi(x_n) \, e^{-S_{int}} \} | 0 \rangle_E^{Free}}{\langle 0 | T\{ e^{-S_{int}} \} | 0 \rangle_E^{Free}}$$

Using $S_{int}^{(E)} \sim -i \, S_{int}^{(M)}$ we get perfect correspondence with the formula for pert. theory (Peskin Ch 4) given the identification

$$\int dt\, H_I = -S_{int}^{(M)} = -\int dt\, L_{int}[\varphi]$$

In the canonical formalism one writes $H_I = -L_{int} = \int d^3x\, \mathcal{L}_{int}$ even without proof for derivative interactions. With the path integral this is trivial.

- Having made contact with the Dyson formula, all knowledge from QM can be used here. Feynman rules for the external states we will not discuss, since it requires to define scattering amplitudes. You will see in Advanced QFT that these emerge from a manipulation of time-ordered correlators.

The path integral suggests a simpler way to compute the Feynman rules for an interaction vertex.

Define the generating functional for the interacting theory as

notation for $\int d^4x\, J(x)\,\varphi(x)$

$$Z[J] \equiv \int D\varphi\; e^{-S_0 - S_{int} + J\cdot\varphi}$$

$$= \int D\varphi \sum_k \frac{1}{k!}\left(-S_{int}[\varphi]\right)^k e^{-S_0 + J\cdot\varphi}$$

$$= \sum_k \frac{1}{k!} \int D\varphi \left(-S_{int}\left[\frac{\partial}{\partial J}\right]\right)^k e^{-S_0 + J\cdot\varphi}$$

What we mean is that any insertion of the field can be replaced by the functional derivative,

$$S_{int} = \int d^4x\, \varphi^4(x)$$

$$S_{int} = \int d^4x\, \frac{\partial}{\partial J(x)}\frac{\partial}{\partial J(x)}\frac{\partial}{\partial J(x)}\frac{\partial}{\partial J(x)}$$

Therefore,

$$Z[J] = \sum_k \frac{1}{k!}\left(-S_{int}\left[\frac{\partial}{\partial J}\right]\right)^k Z_0[J]$$

$$= \exp\left\{-S_{int}\left[\frac{\partial}{\partial J}\right]\right\} Z_0[J]$$

This formal expression has the conceptual beauty of showing how all correlators of the interacting theory, equivalent to $Z[J]$, can be obtained algorithmically from $Z_0[J]$, which is given by

$$Z_0[J] = Z_0[0]\, e^{\frac{1}{2} J\cdot D\cdot J}$$

We can compute the Feynman rule by taking the first term in the series,

$$Z^{(1)}[J] = -S_{int}\left[\frac{\partial}{\partial J}\right] e^{\frac{1}{2} J\cdot D\cdot J}\, Z_0[0]$$

$$= -Z_0[0]\, S_{int}[D\cdot J]\, e^{\frac{1}{2} J\cdot D\cdot J} + \dots$$

This is because the rightmost ∂' gives

$$\frac{\partial}{\partial J} e^{\frac{1}{2} J\cdot D\cdot J} = \int d^4y\, D_{xy} J(y)\, e^{\frac{1}{2} J\cdot D\cdot J} \equiv D\cdot J\, e^{\frac{1}{2} J\cdot D\cdot J}$$

The next ∂' acts on both $e^{\frac{1}{2} J\cdot D\cdot J}$ and $D\cdot J$. The latter case gives

$$\frac{\partial}{\partial J(x)} \int d^4y\, D_{xy} \frac{\partial J(y)}{\partial J(x)} = \int d^4y\, D_{xy}\, \delta^4(x-y) = D_{xx}$$

Which will give terms like

The Feynman rule is obtained by computing the connected component of the L-point correlator, at first order in the S_{int} expansion.

$$\langle 0 | T\{ \varphi(x_1) \cdots \varphi(x_L) \} | 0 \rangle_E^{(1)} = \frac{1}{Z[0]} \frac{\partial}{\partial J(x_1)} \cdots \frac{\partial}{\partial J(x_L)} Z^{(1)}[J] \Big|_{J=0}$$

Then if we "waste" two legs in D_{xx}, we end up with less than L factors of $D \cdot J$ in $Z^{(1)}$.

Given that we eventually set $J=0$, the only nonvanishing term is

$$S_{int}[D \cdot J]\, e^{\frac{1}{2} J \cdot D \cdot J}$$

in $Z^{(1)}$, and when all L ∂'s act on S_{int} in the correlator,

$$\frac{\delta}{\delta J(x)} S_{int}[D \cdot J] = \int d^4y \frac{\delta S_{int}[\varphi]}{\delta \varphi(y)} \underbrace{\frac{\delta \varphi(y)}{\delta J(x)}}_{D_{yx}}$$

$$\langle 0 | T\{ \varphi(x_1) \cdots \varphi(x_L) \} | 0 \rangle = - \int d^4y_1 \cdots d^4y_L \frac{\partial^L S_{int}[\varphi]}{\delta \varphi(y_1) \cdots \delta \varphi(y_L)} D_{y_1 x_1} \cdots D_{y_L x_L}$$

So the Feynman rule is the functional derivative of the interaction

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$$

$$\hookrightarrow S_{int}^{(E)} = \left[-i S_{int} \; ; \; t \to -i\tau\right]$$

$$= \left[-\frac{i\lambda}{4!}\int d^4x\, \varphi^4(x) \; ; \; t \to -i\tau\right]$$

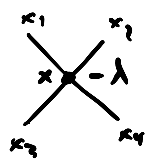
$$= +\frac{\lambda}{4!}\int d^4x\, \varphi^4(x)$$

We get

$$\frac{\partial^4 S_{int}}{\delta\varphi\dots\delta\varphi} = \lambda\int d^4x\, \delta(y_1-x)\cdots\delta(y_4-x)$$

The delta functions are characteristic of local interactions. They allow to perform the φ integrals, with only x, the location of the vertex, remaining.

$$\langle 0|T\cdot\cdot|0\rangle = -\lambda\int d^4x\, D_{xx_1}\cdots D_{xx_4}$$

$=$ 

The Feynman rule is the amputated correlator, i.e. without external legs

$$\left[\,\times\,\right] = -\lambda$$

For the Minkowski correlators, do $x_i^0 = x_i^{E,0} \to i x_i^0$ and $dx^0 \to i\, dx^0$, so

$$\left[\times\right]_{Mink} = +i\left[\times\right]_{Euc.} = -i\lambda$$

We can do derivative interactions without much problem. Writing the interaction in Fourier space,

$$
\begin{aligned}
S_{int}[\varphi] &= \int d^4x\, \partial\cdots\partial\varphi(x)\cdots\partial\varphi(x) \\
&= \int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_L}{(2\pi)^4}\,(-ip_1)\cdots\tilde{\varphi}(p_1)\cdots(-ip_L)\tilde{\varphi}(p_L)\cdot\int d^4x\, e^{-ix\sum p_i} \\
&= \int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_L}{(2\pi)^4}\, I[\tilde{\varphi}]\,(2\pi)^4\,\delta^4\Big(\sum_i p_i\Big)
\end{aligned}
$$

↳ polynomial in $\tilde{\varphi}$ with mom. dep.

The functional derivative on φ is

$$\frac{\delta S_{int}}{\delta\varphi(x)} = \int d^4k\, \underbrace{\frac{\delta\tilde{\varphi}(k)}{\delta\varphi(x)}}_{e^{ikx}}\,\frac{\delta S_{int}[\varphi]}{\delta\tilde{\varphi}(k)}$$

$$\Rightarrow \frac{\delta^L S_{int}}{\delta\varphi(x_1)\cdots\delta\varphi(x_L)} = \int \frac{d^4p_1}{(2\pi)^4}\, d^4k_1\, e^{ik_1x_1}\cdots\frac{d^4p_L}{(2\pi)^4}\, d^4k_L\, e^{ik_Lx_L}\cdot(2\pi)^4\delta^4\Big(\sum_i p_i\Big) \times \frac{\delta^L I[\tilde{\varphi}]}{\delta\tilde{\varphi}(k_1)\cdots\delta\tilde{\varphi}(k_L)}$$

Each y_i integral gives

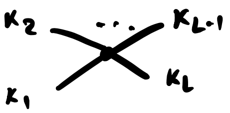
$$\int dy_i\, e^{ik_i y_i} \underbrace{e^{-iq_i y_i}}_{D y_i x_i} = (2\pi)^4 \delta^4(k_i - q_i)$$

So we can do the dq_i propag. integral, getting

$$\langle 0|T\{\varphi(x_1)\cdots\varphi(x_n)\}|0\rangle = \int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_L}{(2\pi)^4}\, e^{ik_1x_1}\cdots e^{ik_Lx_L}\, \frac{1}{\tilde{K}(k_1)}\cdots\frac{1}{\tilde{K}(k_L)}$$

$$\times(-1)\cdot\frac{\delta^L I[\tilde{\varphi}]}{\delta\tilde{\varphi}(k_1)\cdots\delta\tilde{\varphi}(k_L)}\,(2\pi)^4\delta^4\Big(\sum_i k_i\Big)$$

So the general Feynman rule is



$$= -\frac{\partial^L I}{\partial\tilde{\varphi}(k_1)\cdots\partial\tilde{\varphi}(k_L)} = -\frac{\partial^L S_{int}[\tilde{\varphi}]}{\partial\tilde{\varphi}^L}\cdot(2\pi)^{4L}\Big/(2\pi)^4\delta^4\Big(\sum_i k_i\Big)$$

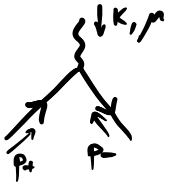
Here k momenta is represented as incoming momentum. This is what you will find by LSZ.

We can easily go to Minkowski, $x_i^0 \to i x_i^0$. Each d^4k integral has the same pole structure as the 2-point case, $1/\tilde{K}$. The vertex has no poles, so $k_0 \to -ik_0$, obtaining Mink-space propag times the vertex at $k_0 \to -ik_0$.

We can compute the interaction in scalar QED.

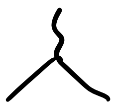
$$\mathcal{L} = \mathcal{L}_{kin} + ie\left(\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi\right) A^\mu + e^2 \phi^\dagger \phi A_\mu A^\mu$$

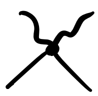
$\hookrightarrow \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 |\phi|^2$



$$= i \frac{\delta^3 S_{int}}{\delta \tilde{\phi}(p_+) \delta \tilde{\phi}(p_-) \delta A_\mu} \Big/ \delta^4(\dots)(2\pi)^{\dots}$$

$$S_{int} = ie \int d^4p_+ \, d^4p_- \, d^4k \left(\tilde{\phi}^\dagger(p_-)(-ip_+)_\mu \tilde{\phi}(p_+) - (-ip_-) \tilde{\phi}^\dagger(p_-) \phi(p_+) \right) \tilde{A}_\mu(k) \cdots$$

$\rightarrow$  $= ie(p_+ - p_-)_\mu$

 $= 2ie^2 \eta_{\mu\nu}$